The Simplex Algorithm

An approach to optimization problems for linear real arithmetic constraints

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Seminar Constraint Logic Programming
University of Osnabrueck
November 11th 2002

Content

- Optimization – what is that?
- The Simplex Algorithm – background
- Simplex form
- Basic feasible solved form / basic feasible solution
- The algorithm
- Initial basic feasible solved form

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- **Optimization – what is that?**
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Optimization – what is that?

- In general

- Optimization problem *(C,f)* with constraint C and objective function f

  e.g.

$$C := X + Y \geq 4$$

Optimization – what is that?

Objective function f:

- expression over variables V in constraint C
- evaluates to a real number
- e.g.

$$f := X^2 + Y^2$$

Optimization – what is that?

a valuation θ (substituting variables by values):

$$\theta = \{ X_1 \leftarrow v_1, X_2 \leftarrow v_2, \ldots , X_n \leftarrow v_n \}$$

$X_{1\ldots n}$ variables

$v_{1\ldots n}$ values

solution of objective function using θ:

$$f(\theta) := f(v_1, v_2, \ldots, v_n)$$

Optimization – what is that?

preferred valuations:

- valuation θ is *preferred* to valuation θ',
  if $f(\theta) < f(\theta')$

optimal solution:

- θ is *optimal,*
  if $f(\theta) < f(\theta')$ for all solutions $\theta' \neq \theta$
  (there is no solution that is preferred to θ)

Optimization – what is that?

Do all problems have an optimal solution?

$$X \leq 7 \wedge X \geq 49$$

$$X \leq 77 \text{ with } f(X) = X$$

Optimization Example

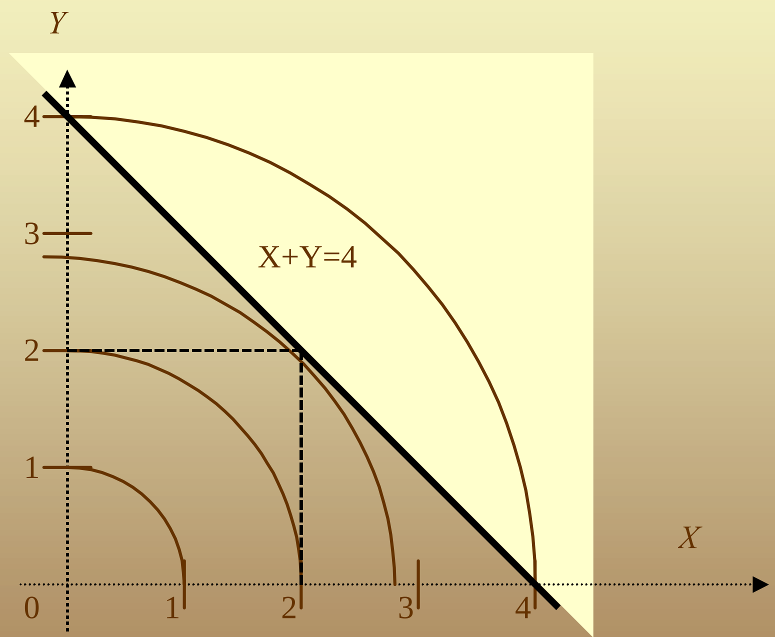
An optimization problem

$$(C \equiv X + Y \geq 4, f \equiv X^2 + Y^2)$$

Find the closest point to the origin satisfying the *C*.

Some solutions and *f* value

$\{X = 0, Y = 4\}$	16
$\{X = 3, Y = 3\}$	18
$\{X = 2, Y = 2\}$	8



Optimal solution

$$\{X = 2, Y = 2\}$$

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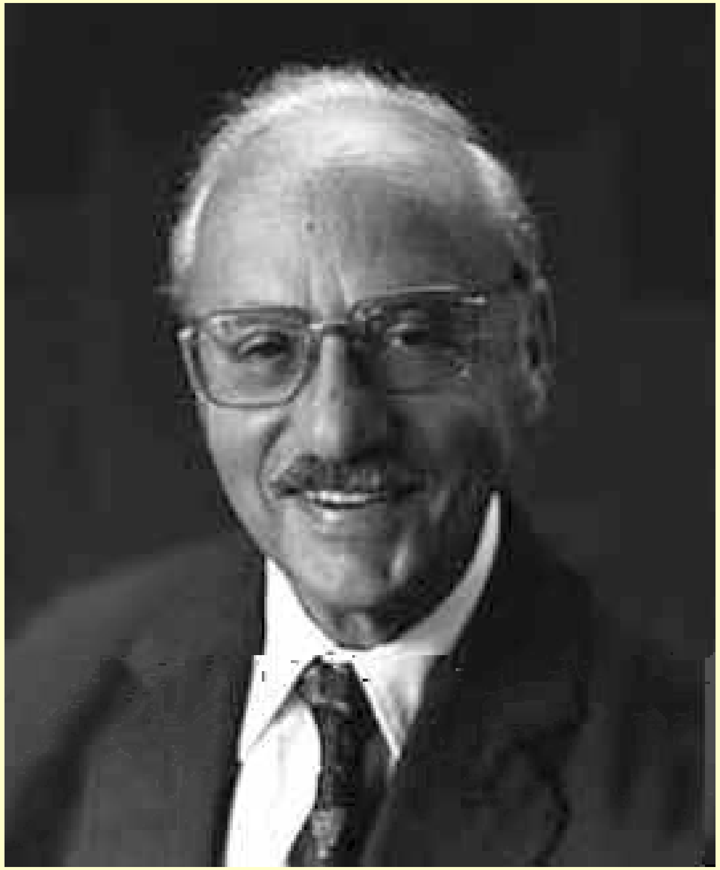
Background



- George Dantzig
- born 8.11.1914, Portland
- invented "Simplex Method of Optimisation" in 1947
- this grew out of his work with the USAF

Background

- originates from planning tasks:
  - plans or schedules for training
  - logistical supply
  - deployment of men
- has in practice usually polynomial cost

Quotes

Eugene Lawler (1980):

[Linear programming] *is used to allocate resources, plan production, schedule workers, plan investment portfolios and formulate marketing (and military) strategies. The versatility and economic impact of linear programming in today's industrial world is truly awesome.*

Quotes

Dantzig I:

The tremendous power of the simplex method is a constant surprise to me.

Dantzig II:

... it is interesting to note that the original problem that started my research is still outstanding - namely the problem of planning or scheduling dynamically over time, particularly planning dynamically under uncertainty. If such a problem could be successfully solved it could eventually through better planning contribute to the well- being and stability of the world.

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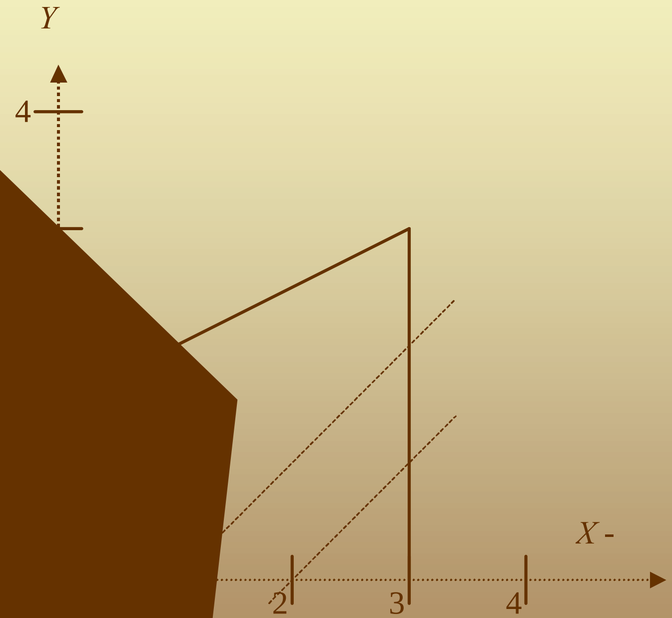
[redacted] *example*



Simplex form

- *(C,f)* is in *simplex form*,
  if C has the form $C_E \wedge C_I$
- C_E is a conjunction of linear arithmetic equations
- C_I is a term that constrains all variables in C to be ≥ 0

Simplex form

allowed conversions to get simplex form:

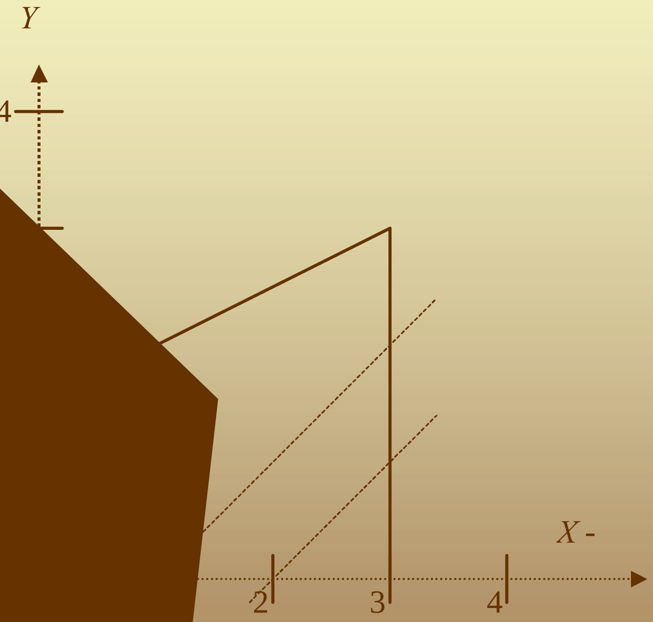
– X not constrained to be non-negative:

$$X = X^+ - X^- \quad \text{with } X^+ \geq 0 \text{ and } X^- \geq 0$$

– inequality e ≤ r (e=expression and r=number)

$$e \leq r \Leftrightarrow e + S = r \quad \text{with } S \geq 0$$

example



An equivalent simplex form is:

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Basic feasible solved form

feasible = practicable, able to be carried out (durchführbar, anwendbar)

- a simplex form optimization problem is in *basic feasible solved form*,

  if all equations in C_E (of the simplex form) have the form:

  $$X_0 = b + a_1 X_1 + \cdots + a_n X_n$$

Basic feasible solved form

$$X_0 = b + a_1 X_1 + \cdots + a_n X_n$$

- X_0 is called basic variable, does not occur anywhere else (neither in objective function)
- $X_{1\ldots n}$ are parameters
- b, $a_{1\ldots n}$ are constants
- $b \geq 0$

Basic feasible solution

$$X_0 = b + a_1 X_1 + \cdots + a_n X_n$$

- corresponding basic feasible solution to a basic feasible solved form:
  - setting each $X_{1...n} = 0$

$$\Rightarrow X_0 = b$$


X -2

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The algorithm

- idea: optimal solution has to be in one of the vertices
- so: go from one vertex to the preferred next vertex
- end: if there is no preferred vertex, the actual has to be the optimal solution

The algorithm

in other words:

- take a basic feasible solved form
- look for an "adjacent" basic feasible solved form whose basic feasible solution decreases the value of the objective function
- if there is no such adjacent basic feasible solved form, then the optimum has been found

The algorithm

- adjacent ≡ just one single pivot
- pivoting ≡ move one variable out of basic variables (≡ *exit variable*) and another in (≡ *entry variable*)

$$X = 49 - 7Y + 21Z$$

$$Y = 7 + 3Z - \tfrac{1}{7}X$$

The algorithm

$$f = e + \sum_{j=1}^{m} d_j Y_j$$

Problem:

Which variables should be exiting resp. entering?

$$\bigwedge_{i=1}^{n} \left(X_i = b_i + \sum_{j=1}^{m} a_{ij} Y_j \right) \quad \wedge$$

$$\bigwedge_{i=1}^{n} (X_i \geq 0) \quad \wedge \quad \bigwedge_{j=1}^{m} (Y_j \geq 0)$$

Entering Variable:

– choose one Y_J with $d_J < 0$

$\Rightarrow$ pivoting on this Y_J can only decrease f (see next slide)

– no such $Y_J \leftrightarrow$ optimum has been found

Why pivoting on a Y_j with $d_j<0$ decreases objective function f

$$f = e + d_1 Y_1 + \ldots + d_J Y_J$$

- looking at the basic feasible solution (bfs) every parametric variable (Y_j) is set to 0
- pivoting on such a variable (var. becomes basic) leads to an increase of this variable in the bfs: $Y_j \geq 0$

$\Rightarrow$ a Y_j with negative d_j decreases f

The algorithm

Exiting variable:

- we have to maintain basic feasible solved form

$\Rightarrow$ all b_i's have to be ≥ 0

$\Rightarrow$ choose a X_i so that $-b_I/a_{IJ}$ is a minimum of:

$$M = \left\{ \frac{-b_i}{a_{iJ}} \middle| a_{iJ} < 0 \text{ and } 1 \leq i \leq n \right\}$$

- M={Ø} $\leftrightarrow$ optimization problem unbounded

Simplex Example

minimize $10 - Y - Z$ subject to

$$X = 3 - Y \wedge$$
$$T = 4 + 2Y - 2Z \wedge$$
$$X \geq 0 \wedge Y \geq 0 \wedge Z \geq 0 \wedge T \geq 0$$

Choose variable *Y*, the first eqn is only one with neg. coeff $Y = 3 - X$

minimize $7 + X - Z$ subject to

$$Y = 3 - X \wedge$$
$$T = 10 - 2X - 2Z \wedge$$
$$X \geq 0 \wedge Y \geq 0 \wedge Z \geq 0 \wedge T \geq 0$$

Choose variable *Z*, the 2nd eqn is only one with neg. coeff $Z = 5 - X - 0.5T$

minimize $2 + 2X + 0.5T$ subject to

$$Y = 3 - X \wedge$$
$$Z = 5 - X - 0.5T \wedge$$
$$X \geq 0 \wedge Y \geq 0 \wedge Z \geq 0 \wedge T \geq 0$$

No variable can be chosen, optimal value 2 is found

The algorithm

starting from a problem in bfs form

repeat

Choose a variable y with negative coefficient in the obj. func.

Find the equation $x = b + cy + \ldots$ where $c<0$ and $-b/c$ is minimal

Rewrite this equation with y the subject $y = -b/c + 1/c\ x + \ldots$

Substitute $-b/c + 1/c\ x + \ldots$ for y in all other eqns and obj. func.

until no such variable y exists or no such equation exists

if no such y exists optimum is found

else there is no optimal solution

The example

Basic feasible solution form: circle

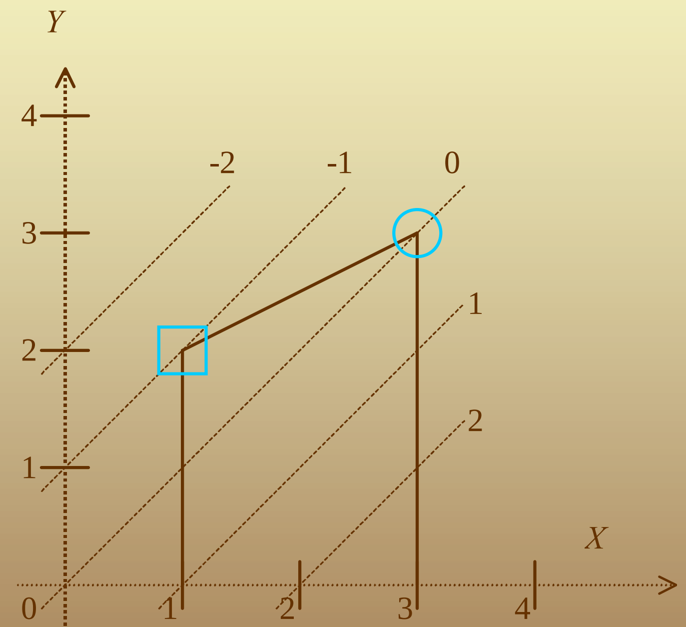
minimize $0 + 0.5S_1 - 0.5S_3$ subject to

$$Y = 3 - 0.5S_1 - 0.5S_3 \quad \wedge$$
$$S_2 = 2 - S_3 \quad \wedge$$
$$X = 3 - S_3 \quad \wedge$$

Choose S3, replace using 2nd eq

minimize $-1 + 0.5S_1 + 0.5S_2$ subject to

$$Y = 2 - 0.5S_1 + 0.5S_2 \quad \wedge$$
$$S_3 = 2 - S_2 \quad \wedge$$
$$X = 1 + S_2 \quad \wedge$$



Optimal solution: box

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Initial basic feasible solved form

idea:

- solve a different optimization problem

this optimization problem should have an initial basic feasible solved form, which:

- can be found trivially
- has an optimal solution that leads to an initial basic feasible solved form of the original problem

Initial basic feasible solved form

$$f = e + \sum_{j=1}^{m} d_j x_j : \quad \bigwedge_{i=1}^{n} \left(\sum_{j=1}^{m} a_{ij} x_j = b_i \right) \quad \wedge \quad \bigwedge_{j=1}^{m} \left(x_j \geq 0 \right)$$

add artificial variables and minimize on them:

$$f = \sum_{i=1}^{n} z_i : \quad \bigwedge_{i=1}^{n} \left(z_i = b_i - \sum_{j=1}^{m} a_{ij} x_j \right) \quad \wedge$$

$$\bigwedge_{j=1}^{m} \left(x_j \geq 0 \right) \quad \wedge \quad \bigwedge_{i=1}^{n} \left(z_i \geq 0 \right)$$

Initial basic feasible solved form

to get basic feasible solved form:

$$f = \sum_{i=1}^{n} z_i = \sum_{i=1}^{n} \left(b_i - \sum_{j=1}^{m} a_{ij} x_j \right)$$

$\Rightarrow$ solve this problem

Initial basic feasible solved form

possible outcomes:

- $(f > 0) \leftrightarrow$ original problem unsatisfiable
- $(f = 0) \wedge (z_{i \ldots n}$ parametric$) \leftrightarrow$ got a basic feasible solved form for original problem
- $(f = 0) \wedge \neg(z_{i \ldots n}$ parametric$) \leftrightarrow z_i$ must occur in such an equation:

$$z = 0 + \sum_{i=1}^{n} d'_i \, z_i + \sum_{j=1}^{m} a'_j \, x_j$$

Such an equation is no problem, because

$$z = 0 + \sum_{i=1}^{n} d'_i \, z_i + \sum_{j=1}^{m} a'_j \, x_j$$

- if all $a'_j = 0 \rightarrow$ equation is redundant
- if one $a'_j \neq 0 \rightarrow$ use according x_j for pivoting z out of basic variables (this maintains basic feasible solved form since z = 0 +…)

$\Rightarrow$ all z become parametric

The example

$$\text{minimize } X - Y \text{ subject to}$$

$$Y \geq 0 \wedge$$

$$X \geq 1 \wedge$$

$$X \leq 3 \wedge$$

$$2Y \leq X + 3$$

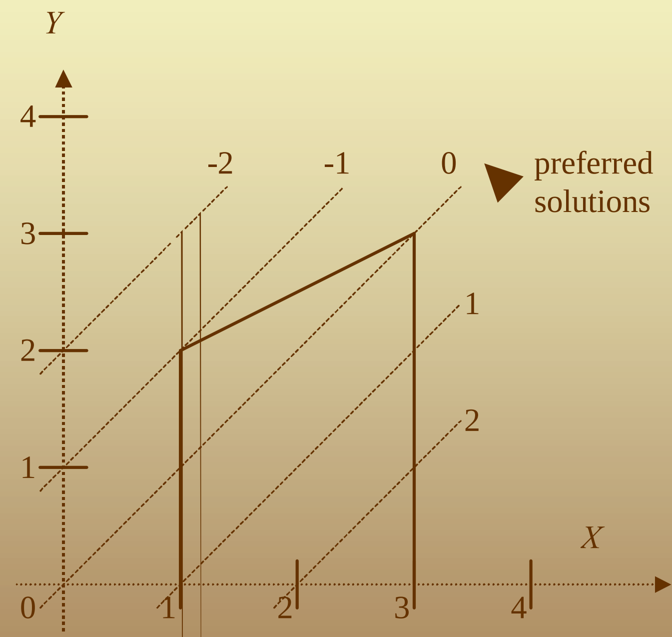
An equivalent simplex form is:

$$X \quad - S \quad = 1 \wedge$$

$$X \quad + S \quad = 3 \wedge$$

$$-X + 2Y + S = 3 \wedge$$

$$X \geq 0 \wedge Y \geq 0 \wedge S \geq 0 \wedge S \geq 0 \wedge S \geq 0$$



An optimization problem showing contours of the objective function

The example

Original simplex form equations

$$X \qquad - S_2 = 1 \wedge$$

$$X \qquad + S_3 = 3 \wedge$$

$$-X + 2Y - S_1 = 3$$

With artificial vars in bfs form:

$$A_1 = 1 - X + S_2 \qquad \wedge$$

$$A_2 = 3 - X - S_3 \qquad \wedge$$

$$A_3 = 3 + X - 2Y - S_1$$

Objective function: minimize

$$A_1 + A_2 + A_3$$

$$= 7 - X - 2Y - S_1 + S_2 - S_3$$

The example

Problem after minimization of objective function

$$\text{minimize } A_1 + A_2 + A_3 \text{ subject to}$$

$$\begin{array}{lllllllll} Y & = & 3 & -0.5S_1 & -0.5S_3 & & -0.5A_2 & -0.5A_3 & \wedge \\ S_2 & = & 2 & & -S_3 & +A_1 & -A_2 & & \wedge \\ X & = & 3 & & -S_3 & & -A_2 & & \end{array}$$

Removing the artificial variables, the original problem

$$\begin{array}{llllll} Y = & 3 & -0.5S_1 & -0.5S_3 & \wedge \\ S_2 = & 2 & & -S_3 & \wedge \\ X = & 3 & & -S_3 & \wedge \end{array}$$

Simplex solver

finding a basic feasible solution is exactly a constraint satisfaction problem

$\Rightarrow$ efficient constraint solver for linear inequalities

Cycling

Problem:

– if for one of the basic variables is valid: $X_i = 0 + \ldots$, a pivot could be performed which does not change the corresponding basic feasible solution

$\Rightarrow$ danger of pivoting back

Solution:

– use e.g. Bland's anti-cycling rule (always select candidate with smallest index: x_2 instead of x_4)

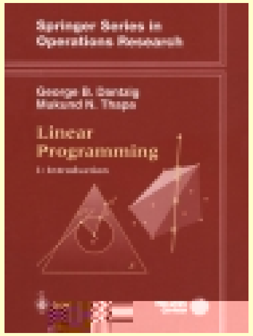
Summary

We have seen that optimisations of linear real arithmetic constraints play an important role in many applications.

The Simplex Method which was introduced here provides a very efficient algorithm to determine whether there exists an optimal solution to linear real arithmetic constraints and if there exists one, to compute it.

Literature

- books:



George B. Dantzig, Mukund N. Thapa

"Linear Programming I: Introduction"

Springer Verlag

Kim Marriott & Peter J. Stuckey

"Programming with Constraints: An Introduction"

MIT Press

Literature

- examples are taken from a presentation of Marriott & Stuckey and could be accessed via internet:

  http://www.cs.mu.oz.au/~pjs/book/course.html

- this presentation in the net:

  http://www-lehre.inf.uos.de/~sbitzer/clp